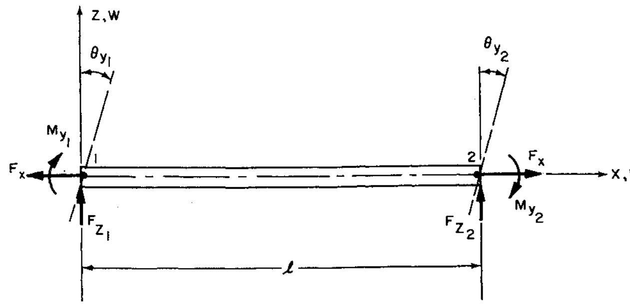

Fig. 1 Uniform beam segment

0, $F_x = -F_{x_{cr}}$, and $l = L/2$ in Eqs. (17), leading to the characteristic equation

$$\left(\frac{8EI}{L} - \frac{F_{x_{cr}}L}{15}\right)\left(\frac{96EI}{L^3} - \frac{12F_{x_{cr}}}{5L}\right) - \left(\frac{24EI}{L^2} - \frac{F_{x_{cr}}}{10}\right)^2 = 0 \quad (18)$$

Solving, one has $F_{x_{cr}} = 9.94(EI/L^2)$, which is only 0.752% different from the exact result, $\pi^2(EI/L^2)$. Reference 1 demonstrates a 10% error in a three-element solution.

It is possible to formulate the exact $[k_f] - [k_h]$ matrix through use of the proper beam column shape and Eq. (11) or by other means. The individual terms, being complicated functions of sines and cosines (or sinh and cosh), are not evaluated as easily as the terms derived in the foregoing. Furthermore, the foregoing technique applies equally well to plate elements in flexure for which exact displacement shapes cannot be found.

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Minimum Structural Mass for a Magnetic Radiation Shield

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STRONG magnetic fields produced by superconducting coils have been suggested as a means of providing shielding in space against intense charged-particle radiation.[1–3] Estimates have shown that the mass of a magnetic shield will be less than that of a bulk shield designed to protect an equivalent volume of the vehicle against energetic protons of about 1 bev or more. These estimates have shown also that most of the mass in a magnetic shield will be contained in the structure required to support the superconducting coil against the large forces exerted upon it by its own magnetic field. It will be shown that a large reduction in the mass can be obtained by suitable design of the supporting structure. General limitations on the minimum structural mass associated with intense field producing coils also will be given.

Specific formulas will be developed for the structural mass required to support a circular cylindrical current sheet of arbitrary length L and radius R. This model is a good approximation for a solenoidal coil for which the thickness of the windings is small in comparison with the radius. The results of this model for $L \gg R$ also are applicable directly to a torus, the minor radius of which is small compared with its major radius. Similarly, the results for $L \ll R$ can be applied to a coil consisting of a single circular turn.

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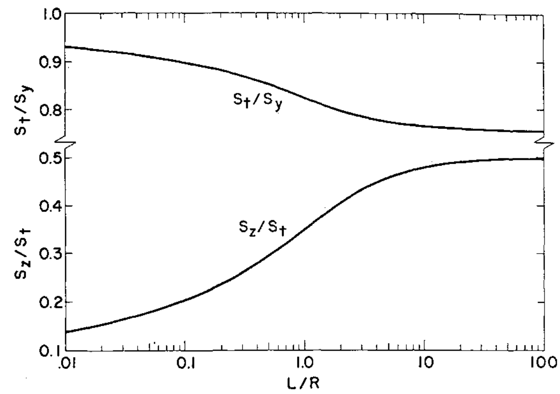

Fig. 1 Variation of stress ratios S_z/S_t and S_t/S_y with L/R

Single Cylindrical Structure

Suppose that for support the solenoid is encased in a simple cylindrical structure C_1, the thickness t of which is small compared with R. The force on each element of current in the solenoid can be resolved into a component directed radially outward and a component directed toward the equatorial plane of the solenoid. Consequently, the forces on the structure can be resolved into an (average) radial pressure P tending to expand the structure and an axial force F_z tending to compress the structure. The radial pressure will be balanced by an (average) azimuthal tensile stress S_t in the structure. Thus, one has approximately

$$RP = tS_t \quad (1)$$

The axial force F_z will be balanced by an (average) compressive axial stress S_z in the structure; hence

$$F_z = 2\pi RtS_z \quad (2)$$

There also will be a radial compressive stress S_r in the struc-

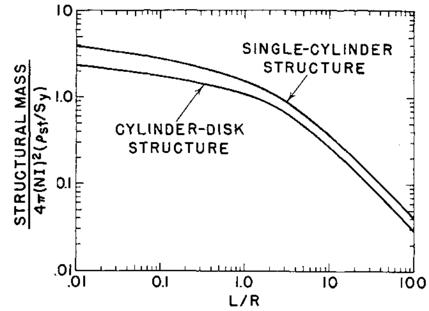

Fig. 2 Variation of structural mass with coil shape L/R

ture, of the order of P, which may be neglected in comparison with S_t. This approximation, which permits one to assume *thin* structural configurations, depends upon the fact that the pressure associated with the maximum critical field of present superconductors is still only a few percent of the yield stress of good structural alloys. With the neglect of S_r, the other two stresses can be related[4] to the yield stress S_y:

$$(S_t/S_y)^2 = [1 + (S_z/S_t) + (S_z/S_t)^2]^{-1} \quad (3)$$

As can be seen, the effect of S_z is to decrease S_t and thereby increase the structural mass.

To obtain the stresses, P and F_z now are expressed in terms of the magnetic energy of the coil:

$$E_{\text{mag}} = \frac{4}{3}\pi(NI)^2 R \left\{\frac{1}{k}[K(k) - E(k)] + \frac{k}{1-k^2}[E(k) - k]\right\} \quad (4)$$

where NI is the total current in abampere-turns,

$$k^2 = 4R^2/(4R^2 + L^2)$$

and $K(k)$ and $E(k)$ represent the complete elliptic integrals of the first and second kind. Using (1) and (4), one has for P

$$2\pi RLP = \frac{\partial E_{\text{mag}}}{\partial R} = 2\pi LtS_t = 4\pi(NI)^2 \frac{k}{1-k^2}[E(k) - k] \quad (5)$$

Similarly, for F_z, one has

$$F_z = -\frac{\partial E_{\text{mag}}}{\partial L} = 2\pi RtS_z = \frac{2}{3}\pi(NI)^2 \left\{\frac{2k^2}{(1-k^2)^{3/2}}[E(k) - k] - \frac{1}{(1-k^2)^{1/2}}[K(k) - E(k)]\right\} \quad (6)$$

From (5) and (6), one obtains the ratio

$$\frac{S_z}{S_t} = \frac{1}{3}\left[2 - \frac{1-k^2}{k^2}\frac{K(k) - E(k)}{E(k) - k}\right] \quad (7)$$

which when substituted into (3) gives S_t and S_z in terms of S_y. The variations of S_z/S_t and S_t/S_y with L/R are shown in Fig. 1. Note that for $L \gg R$, S_z/S_t approaches $\frac{1}{2}$, and, for $L \ll R$, it approaches $[\ln(8R/L) + \frac{1}{2}]^{-1}$.

Finally, t is obtained from either (5) or (6), and the volume $2\pi RLt$ of the cylindrical structure is determined. Multiplying this by ρ_{st}, the density of the structural material, one has for the mass of the structure

$$m_{st} = 4\pi(NI)^2 \frac{\rho_{st}}{S_y}\left[1 + \frac{S_z}{S_t} + \frac{S_z^2}{S_t^2}\right]^{1/2} \frac{k}{1-k^2}[E(k) - k] \quad (8)$$

The variation of m_{st} with L/R is given in Fig. 2. For a long coil, m_{st} approaches $2\pi(NI)^2(\rho_{st}/S_y)\,7^{1/2}\pi(R/L)$. For a very flat coil, m_{st} is approximately $2\pi(NI)^2(\rho_{st}/S_y)[\ln(8R/L) + 1]$. Figure 2 also may be taken as a plot of m_{st} vs L for fixed R and fixed magnetic moment πR^2NI of the coil.

Cylinder-Disk Combination Structure

Consider now a structural configuration consisting of a thinner cylinder (C_2) encasing the coil windings and augmented on the inside by a series of thin parallel disks (D) spread out along the coil axis. The cylinder C_2 is designed to support the coil only against the axial compressive force F_z, whereas the disks are used to resist the radial expansion of the coil.

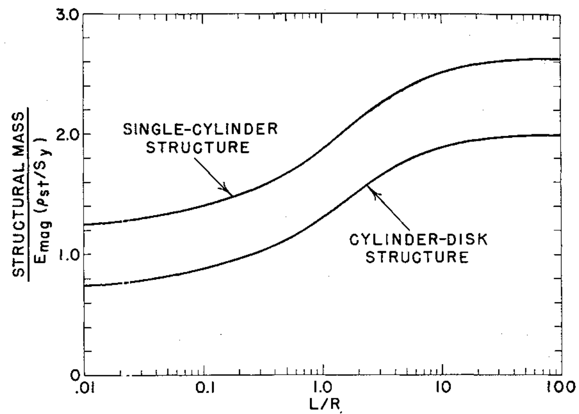

Fig. 3 Comparison of structural mass and magnetic energy of coil

To get the thickness t_c of the cylinder C_2, one has a relation similar to (6), with t replaced by t_c and S_z replaced by S_y. This assumes that S_y is also the maximum compressive force.[4] One thus obtains for the volume or mass ratio of the two cylinders

$$(\text{mass of } C_2)/(\text{mass of } C_1) = t_c/t = S_z/S_y \quad (9)$$

To determine the overall thickness t_d of the disks, take S_y for the radial tensile stress in the disks and set

$$2\pi RLP = 2\pi Rt_dS_y \quad (10)$$

Comparing this with (1), one finds for the mass of the disks

$$(\text{mass of } D)/(\text{mass of } C_1) = \pi R^2t_d/2\pi RLt = S_t/2S_y \quad (11)$$

Adding (10) and (11) and using (3), the mass of the cylinder-disk structure $\bar{m}_{st}$ is obtained in terms of the mass of the single cylinder:

$$\bar{m}_{st} = [1 - \tfrac{3}{4}(S_t/S_y)^2]^{1/2}m_{st} \quad (12)$$

The variation of $\bar{m}_{st}$ with L/R also is shown in Fig. 2. It is seen that $\bar{m}_{st}/m_{st}$ is less than unity for all values of L and R. For $L \gg R$, it is approximately $(\frac{4}{7})^{1/2}$, a reduction of almost one fourth. For $L \ll R$, $\bar{m}_{st}/m_{st}$ approaches $\frac{1}{2}$.

Limitation on Minimum Mass

The distribution of magnetic energy determines the forces on the coil, which in turn determine the stresses and hence the mass of any chosen structural configuration. A lower limit on the mass now is obtained in terms of the coil magnetic energy E_{mag}. For this one starts with a theorem by Longmire and Petschek (presented by Levy in Ref. 5) which relates the energy to the volume integral of the sum of the principal stresses:

$$E_{\text{mag}} = \iiint \left(\sum_{i=1}^{3} S_i\right) dV \quad (13)$$

where a stress is taken positive if tensile and negative if compressive. Equation (13) is obtained by expressing the forces on a small volume element in terms of the material stress tensor and magnetic stress tensor and integrating these with the help of Green's theorem over all space.

Let V_{st} be the total volume of the structure and $\langle\Sigma S_i\rangle$ the volume average of the sum of the principal stresses. As before, stresses in the superconducting material will be neglected, and the integration will be restricted to the volume of the structure. Then (13) yields

$$m_{st} = \rho_{st}V_{st} = \rho_{st}\,E_{\text{mag}}/\langle\Sigma S_i\rangle \quad (14)$$

It can be seen that the minimum mass will be obtained when $\langle \Sigma S_i \rangle$ is a maximum. This will occur when the structure is under maximum isotropic tensile stress S_0, that is, when $S_1 = S_2 = S_3 = S_0$, provided that S_0 is not much less than S_y. It follows then that

$$m_{st} \geqslant \tfrac{1}{3}\rho_{st}\, E_{\text{mag}}/S_0 \tag{15}$$

This minimum may be difficult to achieve for any practical coil design. It perhaps may be approached in a coil design that produces a quadrapole-like field at a large distance, but such a coil does not appear attractive for shielding purposes.

The minimum mass for a structure consisting of thin plates or shells will occur when each section is under maximum biaxial tensile stress, that is, when each principal stress in the plane of the shell equals S_y, and the stress normal to the shell may be neglected. In this case, one has

$$m_{st} \geqslant \tfrac{1}{2}\rho_{st} E_{\text{mag}}/S_y \tag{16}$$

The minimum in (16) is approached for a very flat (or single-turn) solenoid supported by the disk-like structure. In this example, the mass is mostly in the disks, which are under tensile stress S_y in both the radial and azimuthal directions.

In a previous result presented by Levy,[5] a minimum mass twice that of (16) was obtained by assuming that in any element of volume each principal stress is supported by an individual structural element. The possibility of a structural member bearing more than one stress, in particular, more than one tensile stress, apparently was not considered.

The ratio of the structural mass to $\rho_{st}E_{\text{mag}}/S_y$ is shown in Fig. 3 for the single-cylinder and disk-cylinder structures. In accordance with (14), as L/R approaches zero the ratios approach unity and $\frac{1}{2}$, respectively. The increase in these ratios with increasing L/R can be attributed to the additional mass required to support the compressive force F_z. The lower mass of the disk-cylinder combination results from the replacement of some of the single tensile-stressed material in the single-cylinder by double tensile-stressed material in the disks. If, instead of the disks, (single tensile-stressed) spokes had been used, no mass savings would have been realized. The mass savings obtained by replacing uniaxially stressed structural elements (e.g., spokes or rings) with biaxially stressed elements (e.g., disks or cones) may be useful in other applications.

Concluding Comments

With the decrease in structural mass gained by using biaxially stressed structural elements, additional savings in the overall mass by reducing the amount of superconducting wire used or by reducing the mass of the cryogenic components becomes more important. The development of improved structural materials will enhance the magnetic shield concept further.

To complete the design of a magnetic shield, more detailed information concerning the variation of the shielded volume with coil shape and magnetic moment is required.

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Experimental Unsteady Airfoil Lift and Moment Coefficients for Low Values of Reduced Velocity

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A SOMEWHAT open question of long standing relates to the accuracy of the classical aerodynamic theory for oscillations of a two-dimensional airfoil at low values of reduced velocity or, alternatively, at high values of reduced frequency, k. The situation has been summarized by Bisplinghoff et al.,[1] noting the difficulties encountered by various investigators in obtaining experimental data for $1/k < 2.0$, and particularly for $1/k < 1.0$. The various experimental data show generally excellent agreement with theory at high values of reduced velocity ($1/k > 2.5$), except for the lift and moment due to pitching; however, at lower values of reduced velocity ($0.8 < 1/k < 2.0$) several of the coefficients show increasingly poor agreement between theory and experiment.

Interest in this question has been maintained over a number of years not only for its intrinsic importance in the development of unsteady aerodynamic theory, but because a steady increase in available data on flutter of foils in dense media, more typical of hydrofoil boat applications, has revealed the apparent existence of serious discrepancies between calculated and measured flutter speeds.[2] Because of the low values of reduced velocity generally involved, some controversy has developed as to the nature of these discrepancies and the implications regarding the classical theory; recent theoretical and experimental studies[3] have not clarified the picture. An effective answer to this problem probably will depend primarily upon the availability of fundamental experimental data on the unsteady aerodynamic coefficients.

The authors presently are engaged in a study of hydroelastic characteristics of submerged lifting surfaces for hydrofoil applications and have acquired experimental data on oscillatory lift and moment distributions on a rectangular cantilever foil of aspect ratio 5 in water under fully wetted flow conditions.[4] These data at a low reduced velocity are of such nature that they may be correlated directly with similar data obtained during wind-tunnel tests at higher reduced velocities,[5] at essentially the same Reynolds numbers. Almost identical experimental techniques and procedures were employed in both programs.

Figures 1–4 present plots of the oscillatory lift and moment coefficients due to bending and torsion. In each figure, experimental data from both wind-tunnel and water-towing tank tests at 50% semispan are compared with the classical two-dimensional theory. The coefficients are given in terms of magnitude and phase angle vs reduced velocity, $1/k$, where the following notation applies:

L_{hh} = nondimensional oscillatory lift coefficient due to bending, positive downward = L_h

$L_{h\alpha}$ = nondimensional oscillatory lift coefficient due to torsion about the quarter-chord, positive downward = L_α

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